Deptt- MATHEMATICS

Topic- Indefinite Integration

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 1

(e) Integration of trigonometric functions :

(i) $\int \frac{dx}{a+b\sin^2 x}$ OR $\int \frac{dx}{a+b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Divide N^r & D^r by $\cos^2 x$ & put $\tan x = t$.

Illustration 17 : Evaluate : $\int \frac{dx}{2+\sin^2 x}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$I = \int \frac{\sec^2 x dx}{2\sec^2 x+\tan^2 x} = \int \frac{\sec^2 x dx}{2+3\tan^2 x}$$

Let $\sqrt{3}\tan x = t$ $\therefore$ $\sqrt{3}\sec^2 x\, dx = dt$

So $I = \frac{1}{\sqrt{3}}\int \frac{dt}{2+t^2} = \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)+c$ **Ans.**

Illustration 18 : Evaluate : $\int \frac{dx}{(2\sin x+3\cos x)^2}$

Solution : Divide numerator and denominator by $\cos^2 x$

$$\therefore \quad I = \int \frac{\sec^2 x dx}{(2\tan x+3)^2}$$

Let $2\tan x+3 = t$, $\therefore$ $2\sec^2 x dx = dt$

$I = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t}+c = -\frac{1}{2(2\tan x+3)}+c$ **Ans.**

Do yourself -10 :

(i) Evaluate : $\int \frac{dx}{1+4\sin^2 x}$ **(ii)** Evaluate : $\int \frac{dx}{3\sin^2 x+\sin x\cos x+1}$

(ii) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles & put $\tan\frac{x}{2} = t$

In this case $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, $x = 2\tan^{-1}t$; $dx = \frac{2dt}{1+t^2}$

Illustration 19 : Evaluate : $\int \frac{dx}{3\sin x+4\cos x}$

Solution :

$$I = \int \frac{dx}{3\sin x+4\cos x} = \int \frac{dx}{3\left\{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}+4\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}} = \int \frac{\sec^2\frac{x}{2}dx}{4+6\tan\frac{x}{2}-4\tan^2\frac{x}{2}}$$

let $\tan\frac{x}{2} = t$, $\therefore$ $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$

so $I = \int \frac{2dt}{4+6t-4t^2} = \frac{1}{2}\int \frac{dt}{1-\left(t^2-\frac{3}{2}t\right)} = \frac{1}{2}\int \frac{dt}{\frac{25}{16}-\left(t-\frac{3}{4}\right)^2}$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{5}{4}\right)}\ell n\left|\frac{\frac{5}{4}+\left(t-\frac{3}{4}\right)}{\frac{5}{4}-\left(t-\frac{3}{4}\right)}\right|+c = \frac{1}{5}\ell n\left|\frac{1+2\tan\frac{x}{2}}{4-2\tan\frac{x}{2}}\right|+c$$ **Ans.**

Do yourself -11 :

(i) Evaluate : $\int\frac{dx}{3+\sin x}$ **(ii)** Evaluate : $\int\frac{dx}{1+4\sin x+3\cos x}$

(iii) $\int\frac{a\cos x+b\sin x+c}{p\cos x+q\sin x+r}dx$

Express Numerator $(N^r) = \ell(D^r) + m\frac{d}{dx}(D^r) + n$ & proceed.

Illustration 20 : Evaluate : $\int\frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}d\theta$

Solution : Write the Numerator = ℓ(denominator) + m(d.c. of denominator) + n

$\Rightarrow 2 + 3\cos\theta = \ell(\sin\theta+2\cos\theta+3) + m(\cos\theta - 2\sin\theta) + n.$

Comparing the coefficients of $\sin\theta$, $\cos\theta$ and constant terms,

we get $3\ell + n = 2,\quad 2\ell + m = 3,\quad \ell - 2m = 0 \Rightarrow \ell = 6/5,\ m = 3/5$ and $n = -8/5$

Hence $I = \int\frac{6}{5}d\theta+\frac{3}{5}\int\frac{\cos\theta-2\sin\theta}{\sin\theta+2\cos\theta+3}d\theta-\frac{8}{5}\int\frac{d\theta}{\sin\theta+2\cos\theta+3}$

$= \frac{6}{5}\theta+\frac{3}{5}\ell n|\sin\theta+2\cos\theta+3| - \frac{8}{5}I_3$ where $I_3 = \int\frac{d\theta}{\sin\theta+2\cos\theta+3}$

In I_3, put $\tan\frac{\theta}{2}=t \Rightarrow \sec^2\frac{\theta}{2}d\theta = 2dt$

$I_3 = 2\int\frac{dt}{t^2+2t+5}=2\int\frac{dt}{(t+1)^2+2^2} = 2.\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$

Hence $I = \frac{6\theta}{5}+\frac{3}{5}\ell n|\sin\theta+2\cos\theta+3|-\frac{8}{5}\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)+c$ **Ans.**

Do yourself -12 :

(i) Evaluate : $\int\frac{\sin x}{\sin x+\cos x}dx$ **(ii)** Evaluate $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

(iv) $\int\sin^m x\cos^n x\,dx$

Case-I : When m & n $\in$ natural numbers.

* If one of them is odd, then substitute for the term of even power.
* If both are odd, substitute either of the term.
* If both are even, use trigonometric identities to convert integrand into cosines of multiple angles.

Case-II : m + n is a negative even integer.

* In this case the best substitution is tanx = t.

Illustration 21 **:** Evaluate $\int \sin^3 x \cos^5 x \, dx$

Solution **:** Put cos x = t; − sin x dx = dt.

so that $I = -\int (1 - t^2).t^5 dt$

$$= \int (t^7 - t^5) dt = \frac{t^8}{8} - \frac{t^6}{6} = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c$$

Alternate :

Put sin x = t; cos x dx = dt

so that $I = \int t^3 (1 - t^2)^2 dt = \int \left(t^3 - 2t^5 + t^7\right) dt$

$$= \frac{\sin^4 x}{4} - \frac{2\sin^6 x}{6} + \frac{\sin^8 x}{8} + c$$

Note : This problem can also be handled by successive reduction or by trignometric identities.

Illustration 22 **:** Evaluate $\int \sin^2 x \cos^4 x \, dx$

Solution **:** $\int \sin^2 x \cos^4 x dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{\cos 2x + 1}{2}\right)^2 dx = \int \frac{1}{8}(1 - \cos 2x)\left(\cos^2 2x + 2\cos 2x + 1\right) dx$

$$= \frac{1}{8}\int \left(\cos^2 2x + 2\cos 2x + 1 - \cos^3 2x - 2\cos^2 2x - \cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(-\cos^3 2x - \cos^2 2x + \cos 2x + 1\right) dx = -\frac{1}{8}\int \left(\frac{\cos 6x + 3\cos 2x}{4} + \frac{1 + \cos 4x}{2} - \cos 2x - 1\right) dx$$

$$= -\frac{1}{32}\left[\frac{\sin 6x}{6} + \frac{3\sin 2x}{2}\right] - \frac{1}{16}x - \frac{\sin 4x}{64} + \frac{\sin 2x}{16} + \frac{x}{8} + c$$

$$= -\frac{\sin 6x}{192} - \frac{\sin 4x}{64} + \frac{1}{64}\sin 2x + \frac{x}{16} + c$$

Illustration 23 **:** Evaluate $\int \frac{\sqrt{\sin x}}{\cos^{9/2} x} dx$

Solution **:** Let $I = \int \frac{\sin^{1/2} x}{\cos^{9/2} x} dx = \int \frac{dx}{\sin^{-1/2} x \cos^{9/2} x}$

Here $m + n = \frac{1}{2} - \frac{9}{2} = -4$ (negative even integer).

Divide Numerator & Denominator by $\cos^4 x$.

$I = \int \sqrt{\tan x} \sec^4 x \, dx = \int \sqrt{\tan x}(1 + \tan^2 x)\sec^2 x dx$

$= \int \sqrt{t}(1 + t^2) dt$ (using tan x = t)

$$= \frac{2}{3}t^{3/2} + \frac{2}{7}t^{7/2} + c = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + c$$

Do yourself -13 :

(i) Evaluate : $\int \frac{\sin^2 x}{\cos^4 x} dx$ **(ii)** Evaluate : $\int \frac{\sqrt{\sin x} dx}{\cos^{5/2} x}$ **(iii)** Evaluate : $\int \sin^2 x \cos^5 x \, dx$

(f) Integration of Irrational functions :

(i) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$

(ii) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$; $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x=\frac{1}{t}$

Illustration 24 : Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$

Solution : Let, $I=\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$ Put $x+1=t^2 \Rightarrow dx=2tdt$

$$\therefore \quad I=\int \frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}.(2t)dt = 2\int \frac{t^2+1}{t^4+t^2+1}dt = 2\int \frac{1+1/t^2}{t^2+1+1/t^2}dt$$

$$=2\int \frac{1+1/t^2}{(t-1/t)^2+(\sqrt{3})^2}.dt = 2\int \frac{du}{u^2+(\sqrt{3})^2} \qquad \left\{\text{where } u=t-\frac{1}{t}\right\}$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+c$$ **Ans.**

Illustration 25 : Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Solution : Let, $I=\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$ put $x-1=\frac{1}{t} \Rightarrow dx=-1/t^2\,dt$

$$I=\int \frac{-1/t^2dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}} = -\int \frac{dt}{\sqrt{3t^2+3t+1}}$$

$$=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+1/12}}=-\frac{1}{\sqrt{3}}\log\left|(t+1/2)+\sqrt{(t+1/2)^2+1/12}\right|+c$$

$$=-\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+c$$ **Ans.**

Illustration 26 : Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution : Let, $I=\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x=\frac{1}{t}$, So that $dx=\frac{-1}{t^2}dt$

$$\therefore \quad I=\int \frac{-1/t^2dt}{(1+1/t^2)\sqrt{1-1/t^2}}=-\int \frac{tdt}{(t^2+1)\sqrt{t^2-1}}$$

again let, $t^2=u$. So that $2t\,dt=du$.

$$= \frac{-1}{2}\int \frac{du}{(u+1)\sqrt{u-1}}$$ which reduces to the form $\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear so that we put $u - 1 = z^2$ so that $du = 2z\,dz$

$$\therefore \quad I = -\frac{1}{2}\int \frac{2zdz}{(z^2+1+1)\sqrt{z^2}} = -\int \frac{dz}{(z^2+2)}$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right) + c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + c$$ **Ans.**

Do yourself -14 :

(i) Evaluate $\int \frac{x}{(x-3)\sqrt{x+1}}dx$ **(ii)** Evaluate : $\int \frac{dx}{x^2\sqrt{1+x^2}}$

Miscelleneous Illustrations :

Illustration 27 : Evaluate $\int \frac{\cos^4 x dx}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}$

Solution : $I = \int \frac{\cos^4 x}{\sin^3 x\{\sin^5 x + \cos^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cos^4 x}{\sin^6 x\{1 + \cot^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cot^4 x \operatorname{cosec}^2 x dx}{(1+\cot^5 x)^{3/5}}$

Put $1 + \cot^5 x = t$

$5\cot^4 x \operatorname{cosec}^2 x dx = -dt$

$$= -\frac{1}{5}\int \frac{dt}{t^{3/5}} = -\frac{1}{2}t^{2/5} + c = -\frac{1}{2}(1+\cot^5 x)^{2/5} + c$$ **Ans.**

Illustration 28 : $\int \frac{dx}{\cos^6 x + \sin^6 x}$ is equal to -

(A) $\ell n|\tan x - \cot x| + c$

(B) $\ell n|\cot x - \tan x| + c$

(C) $\tan^{-1}(\tan x - \cot x) + c$

(D) $\tan^{-1}(-2\cot 2x) + c$

Solution : Let $I = \int \frac{dx}{\cos^6 x + \sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x}dx = \int \frac{(1+\tan^2 x)^2 \sec^2 x dx}{1+\tan^6 x}$

If $\tan x = p$, then $\sec^2 x\, dx = dp$

$$\Rightarrow \quad I = \int \frac{(1+p^2)^2 dp}{1+p^6} = \int \frac{(1+p^2)}{p^4 - p^2 + 1}dp = \int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2 + \frac{1}{p^2} - 1\right)}dp$$

$$= \int \frac{dk}{k^2+1} = \tan^{-1}(k) + c \qquad \left(\text{where } p - \frac{1}{p} = k, \left(1+\frac{1}{p^2}\right)dp = dk\right)$$

$$= \tan^{-1}\left(p - \frac{1}{p}\right) + c = \tan^{-1}(\tan x - \cot x) + c = \tan^{-1}(-2\cot 2x) + c$$ **Ans. (C,D)**

Illustration 29 : Evaluate : $\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}dx$

Solution : $I = \int \frac{2\sin 2x - \cos x}{6-\cos^2 x - 4\sin x}dx = \int \frac{(4\sin x - 1)\cos x}{6 - (1-\sin^2 x) - 4\sin x}dx = \int \frac{(4\sin x - 1)\cos x}{\sin^2 x - 4\sin x + 5}dx$

Put sin x = t, so that cos x dx = dt.

$$\therefore \quad I = \int \frac{(4t-1)dt}{(t^2-4t+5)} \qquad \text{...... (i)}$$

Now, let $(4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$$2\lambda = 4, \ -4\lambda + \mu = -1 \qquad \text{...... (ii)}$$

$$\lambda = 2, \ \mu = 7$$

$$\therefore \quad I = \int \frac{2(2t-4)+7}{t^2-4t+5}dt \qquad \{\text{using (i) and (ii)}\}$$

$$= 2\int \frac{2t-4}{t^2-4t+5}dt + 7\int \frac{dt}{t^2-4t+5} = 2\log|t^2-4t+5| + 7\int \frac{dt}{t^2-4t+4-4+5}$$

$$= 2\log|t^2-4t+5| + 7\int \frac{dt}{(t-2)^2+(1)^2} = 2\log|t^2-4t+5| + 7 \,.\, \tan^{-1}(t-2) + c$$

$$= 2\log|\sin^2 x - 4\sin x + 5| + 7\tan^{-1}(\sin x - 2) + c.$$ **Ans.**

Illustration 30 : The value of $\int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$, is equal to -

(A) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$

(B) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$

(C) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$

(D) none of these

Solution : Here, $I = \int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$

Put $x = 3\cos 2\theta \quad \Rightarrow \quad dx = -6\sin 2\theta d\theta$

$$= \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta}.\sin^{-1}(\sin\theta).(-6\sin 2\theta)d\theta = -6\int \theta.(2\sin^2\theta)d\theta$$

$$= -6\int \theta(1-\cos 2\theta)d\theta = -6\left\{\frac{\theta^2}{2} - \int \theta\cos 2\theta\, d\theta\right\}$$

$$= -6\left\{\frac{\theta^2}{2} - \left(\theta\frac{\sin 2\theta}{2} - \int 1.\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\right\} = -3\theta^2 + 6\left\{\theta\frac{\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + c$$

$$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + c$$ **Ans. (A)**

Illustration 31 : Evaluate : $\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}dx$